Before reading this article, try this math problem: A tennis club has 1025 members and decides to hold a tournament to select a winner. Every member draws a lot to see who will play whom during the first round. The odd man out receives a bye. Loosers are out; winners draw lots to play the next round with any extra person receiving a bye. This routine continues until there is only one person who remains a winner. **How many matches will have to be played**? If you try to **solve this problem before you read the article**, it will be more meaningful to you.

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MATHEMATICS AS A CREATIVE ART

By P. R. HALMOS

• *This article is based on an informal lecture delivered at the University, of Illinois on December 12, 1967, as part of the celebration of the Centennial year of the University. The author, an alumnus of Illinois, was a Professor of Mathematics at the University of Michigan and was a professor at the University of Hawaii at the time of this publication.*

Do you know any mathematicians--and, if you do, do you know anything about what they do with their time? Most people don't. When I get into conversation with the man next to me in a plane, and he tells me that he is something respectable like a doctor, lawyer, merchant, or dean, I am tempted to say that I am in roofing and siding. If I tell him that I am a mathematician, his most likely reply will he 1 hat he himself could never balance his check book, and it must be fun to be a whiz at math. If my neighbor is an astronomer, a biologist., a chemist, or any other kind of natural of social scientist, I am, if anything, worse off--this man *thinks* he knows what a mathematician is, and he *is* probably wrong. He thinks that I spend my time (or should) converting different orders of magnitude, comparing binomial coefficients and powers of 2, or solving equations involving rates of reactions.

C. P. Snow points to and deplores the existence of two cultures; he worries about the physicist whose idea of modern literature is Dickens, and he chides the poet who cannot state "the second law of thermodynamics. Mathematicians, in converse with well-meaning, intelligent, and educated laymen (do you mind if I refer to all non-mathematicians as laymen?) are much worse off than physicists in converse with poets. It saddens me that educated people don't even know that my subject exists. There is something that they call mathematics, but they neither know how the professionals use that word, nor can they conceive why anybody should do it. It is, to be sure, possible that an intelligent and otherwise educated person doesn't know that Egyptology exists, or haematology, but all you have to tell him is that it does, and he will immediately understand in a rough general way why it should and he will have some empathy with the scholar of the subject who finds it interesting.

Usually when a mathematician lectures, he is a missionary. Whether he is talking over a cup of coffee with a collaborator, lecturing to a graduate class of specialists, teaching a reluctant group of freshmen, or addressing a general audience of laymen--he is still preaching and seeking to make converts. He will state theorems and he will discuss proofs and he will hope that when he is done his audience will know more mathematics than they did before. My aim is different--I am not here to proselyte but to enlighten—I seek not converts but friends. I do not want to teach you what mathematics is, but only *that* it is.

I call my subject mathematics—that's what all my colleagues call it, all over the world—and there, quie possible, is the beginning of confusion. The word covers two disciplines—many more, in reality, but two, at least tow, in the same sense in which Snow speaks of two cultures. In order to have some words with which to refer to the ideas I want to discuss, I offer two temporary and ad hoc neologisms. Mathematics, as the word is customarily used, consists of at least two distinct subjects, and I propose to call them ***mathology*** and ***mathophysics***. Roughly speaking, **mathology** is is what is usually called pure mathematics, and **mathophysics** is called applied mathematics, but the qualifiers are not emotionally strong enough to disguise that they qualify the same noun. If the concatenation of syllables I chose here reminds you of other words, no great harm will be done; the rhymes alluded to are not completely accidental. I originally planned to entitle this lecture something like "Mathematics is an art," or "Mathematics is not a science," and "Mathology is useless." When I am through I hope you will recognize that most of you have known about mathophysics before, only you were probably calling it mathematics; I hope that all of you will recognized the distinction between mathology and mathophysics; and I hope that some of you will be ready to embrace, or at least applaud, or at the very least, recognize mathology as a respectable human endeavor.

In the course of the lecture I'll have to use many analogies (literature, chess, painting), each imperfect by itself, but I hope that in their totality they will serve to delineate what I want delineated. Sometimes in the interest of economy of time, and sometimes doubtless unintentionally, I'll exaggerate; when I'm done, I'll be glad to rescind anything that was inaccurate or that gave offense in any other way.

What Mathematicians Do

AS the first step toward telling you what mathematicians do, let me tell you some of the things they do not do. To begin with, mathematicians have very little to do with numbers. You can no more expect a mathematician to be able to add a column of figures rapidly and correctly than you can expect a painter to draw a straight line or a surgeon to carve a turkey—popular legend attributes such skills to these professions, but popular legend is wrong. There is, to be sure, a part of mathematics called number theory, but even that doesn't deal with numbers in the legendary sense—a number theorist and an adding machine would find very little to talk about. A machine might enjoy proving that $1^2 + 5^2 + 3^2 = 153$, and it might even go to discover that there are only five positive integers with the property that the equation indicates (1, 310, 371, 407), but most mathematicians enjoy and respect the theorem that every positive integer is the sum of not more than four squares, whereas the infinity involved in the word "every" would frighten and paralyze any ordinary office machine, and, in any case, that's probably not the sort of thing that the person who relegates mathematicians to numbers had in mind.

Not even those romantic' objects of latter day science fiction, the giant brains, the computing machines that run our lives these days—not even they are of interest to the mathematicians as such. Some mathematicians are interested in the logical problems involved in the reduction of difficult questions to the sort of moronic baby talk that machines understand: the logical design of computing machines is definitely mathematics. Their construction is not, that's engineering, and their product, be it a payroll, a batch of sorted mail, or a supersonic plane is of no mathematical interest or value.

Mathematics is not numbers or machines; it is also not the determination of the heights of mountains by trigonometry, or compound interest by algebra, or moments of inertia by calculus. Not today it isn't. At one point in history each of those things, and others like them, might have been an important and non-trivial research problem, but once the problem is solved, its repetitive application has as much to do with mathematics as the work of a Western Union messenger boy has to do with Marconi's genius.

There are at least two other things that mathematics isn't; one of them is something it never *was* and the other is something it once included and by now has sloughed off. The first is physics. Some laymen confuse mathematics and theoretical physics and speak, for instance, of Einstein as a great mathematician. There is no doubt that Einstein was a great man, but he was no more a great mathematician than he was a great violinist. He used mathematics to find out facts about the universe, and that he successfully used certain parts of differential geometry for that purpose adds a certain piquancy to the appeal of differential geometry. Withal, relativity theory and differential geometry are not the same thing. Einstein, Schrodinger. Heisenberg, Fermi, Wigner, Feynman—great men all, but not mathematicians; some of them, in fact, strongly anti-mathematical, preach against mathematics, and would regard it as an insult to be called a mathematician.

What once was mathematics remains mathematics always, but it can become so thoroughly worked out, so completely understood, and, in the light of millennia of contributions, with hindsight, so trivial, that mathematicians never again need to or want to spend time on it. The celebrated Greek problems (trisect the angle, square the circle, duplicate the cube) are of this kind, and the irrepressible mathematical amateur to the contrary notwithstanding, mathematicians are no longer trying to solve them. Please understand, it isn't that they have given up. Perhaps you have heard that, according to mathematicians, it is impossible to square a circle, or trisect an angle, and perhaps you have heard or read that, therefore, mathematicians are a pusillanimous chicken-hearted lot, who give up easily, and use their ex-cathedra pronouncements to justify their ignorance. The conclusion may be true, and you may believe it if you like, but the proof is inadequate. The point is a small one but a famous one and one of historical interest: let me digress to discuss it for a moment.

A Short Digression

The problem of trisecting the angle is this: given an angle, construct another one that is just one third as large. The problem is perfectly easy, and several methods for solving it are known. The catch is that the original Greek formulation of the problem is more stringent: it requires construction that uses ruler and compasses only. Even that call be done, and I could show you a perfectly simple method in one minute and could show you that it works in two more minutes. The real difficulty is that the precise formulation of the problem is more stringent still. The precise formulation demands a construction that uses a ruler and compasses only and, moreover, severely restricts how they are to be used; it prohibits, for instance, marking two points on the ruler and using the marked points in further constructions. It takes some careful legalism (or some moderately pedantic mathematics) to formulate really precisely just what was and what wasn't allowed by the Greek rules. The modern angle trisector either doesn't know those rules, or he knows them but thinks that the idea is to get a close approximation, or he knows the rules and knows that an exact solution is required but lets wish be father to the deed and simply makes a mistake. Frequently his attitude is that of the visitor from outer space to golf. (If all you want is to get that little white ball in this little green hole, why don't you just go and put it there?)

Allow me to add a short digression to the digression. I'd like to remind you that when a mathematician says that something is impossible, it doesn't mean that it is very very difficult, beyond his powers, and probably beyond the powers of all humanity (or the foreseeable future. That’s what is often meant when some one says it's impossible to travel at the speed of sound five miles above the surface of the earth, or instantaneously to communicate with someone a thousand miles away, or to tamper with the genetic code so as to produce a race of citizens who are simultaneously intelligent and peace-loving. That’s what is belittled by the classic business braggadocio (the impossible takes a little longer). The mathematical impossible is different: it is more modest and more secure. The mathematical impossible is the logical impossible. When a mathematician says that it is impossible to find a positive number whose sum with 10 is less than 10, he merely reminds us that that's what the words mean (positive, sum, 10, less); when he says that it is impossible to trisect every angle by ruler and compasses, he means exactly the same sort of thing, only the number of technical words involved is large enough and the argument that strings them together is long enough that they fill a book, not just a line.

The Start of Mathematics

No one knows when and where mathematics got started, or how, but it seems reasonable to guess that it emerged from the same primitive physical observations (counting, measuring) with which we all begin our own mathematical insight (ontogeny recapitulates phylogeny). It was probably so in the beginning, and it is true still, that many mathematical ideas originate not from pure thought but from material necessity; many, but probably not all. Almost as soon as a human being finds it necessary to count his sheep (or sooner?) he begins to wonder about numbers and shapes and motions and arrangements—curiosity about such things seems to be as necessary to the human spirit as curiosity about earth, water, fire, and air, and curiosity--sheer pure intellectual curiosity about stars and about life. Numbers and shapes and motions and arrangements, and also thoughts and their order, and concepts such as "property" and “relation"—all such things are the raw material of mathematicians. The technical but basic mathematical concept of “group” is the best humanity can do to understand the intuitive concept of "symmetry" and the people who study topological spaces, and ergodie paths, and oriented graphs are making precise our crude and vague feelings about shapes, and motions, and arrangements.

Why do mathematicians study such things, and why should they? What, in other words, motivates the individual mathematician, and why does society encourage his efforts, at least to the extent of providing him with the training and subsequently the livelihood that, in turn, give him the time he needs to think? There are two answers to each of the two questions: because mathematics is practical and because more and more mathematics is an art. The already existing mathematics has more and more new applications each day, and the rapid growth of desired applications suggests more and more new practical mathematics. At the same time, as the quantity of mathematics grows and the number of people who think about it keeps doubling over and over again, more new concepts need application, more new logical interrelations cry out for study, and understanding, and simplification, and more and more the tree of mathematics bears elaborate and gaudy flowers that are, to many beholders, worth more than the roots from which it all comes and the causes that brought it all into existence.

Mathematics Today

Mathematics is very much alive today. There are more than a thousand journals that publish mathematical articles about. 15,000 to 20,000 mathematical articles are printed every year. The mathematical achievements of the last 100 years are greater in quantity and in quality than those of all previous history. Difficult mathematical problems, which stumped Hilbert, Cantor, or Poincaré, are being solved, explained, and generalized by beardless (and bearded) youths in Berkeley and in Odessa.

Mathematicians sometimes classify themselves and each other as either problem-solvers or theory-creators. The problem-solvers answer yes-or-no questions and discuss the vital special cases and concrete examples that are the flesh and blood of mathematics; the theory creators fit the results into a framework, illuminate it all, and point it in a definite direction—they provide the skeleton and the soul of mathematics. One and the same human being can be both a problem-solver and theory-creator, but, usually, he is mainly one or the other. The problem-solvers make geometric constructions, the theory-creators discuss the foundations of Euclidean geometry; the problem-solvers find out what makes switching diagrams tick, the theory-creators prove representation theorems for Boolean algebra. In both kinds of mathematics and in all fields of mathematics the progress in one generation is breath taking. NO one can call himself a mathematician nowadays who doesn’t have at least a vague idea of homological algebra, differential topology, and functional analysis, and every mathematician is probably somewhat of an expert on at least one of these subjects—and yet when I studied mathematics in the 1930’s none of those phrase had been invented, and the subjects they describe existed in seminal forms only.

Mathematics is abstract thought, mathematics is pure logic, mathematics is creative art. All these statements are wrong, but they are all a little right, and they are all nearer the mark than “mathematics is numbers” or “mathematics is geometric shapes.” For the professional pure mathematician, mathematics is the logical dovetailing of a carefully selected sparse set of assumptions with their surprising conclusions via a conceptually elegant proof. Simplicity, intricacy, and above all, logical analysis are the hallmarks of mathematics.

The mathematician is interested in extreme cases—in this respect he is like the industrial experimenter who breaks light bulbs, tears shirts, and bounces cars on ruts. How widely does a reasoning apply, he wants to know, and what happens when it doesn't? What happens when you weaken one of the assumptions, or under what conditions can you strengthen one of the conclusions? It is the perpetual asking of such questions that makes for broader understanding, better technique, and greater elasticity for future problems.

Mathematics—this may surprise you or shock you some—is never deductive in its creation. The mathematician at work makes vague guesses, visualizes broad generalizations, and jumps to unwarranted conclusions. He arranges and rearranges his ideas, and he becomes convinced of their truth long before he can write down a logical proof. The conviction is not likely to come early--it usually comes after many attempts, many failures, many discouragements, many false starts. It often happens that months of work result in the proof that the method of attack they were based on cannot possibly work, and the process of guessing, visualizing, and conclusion-jumping begins again. A reformulation is needed—and—and this too may surprise you--more experimental work is needed. To be sure, by "experimental work" I do not mean test tubes and cyclotrons. I mean thought-experiments. When a mathematician wants to prove a theorem about an infinite-dimensional Hilbert space, he examines its finite-dimensional analogue, he looks in detail at the 2- and 3-dimensional cases, he often tries out a particular numerical case, and he hopes that he will gain thereby an insight that pure definition-juggling has not yielded. The deductive stage, writing the result down, and writing down its rigorous proof are relatively trivial once the real insight arrives; it is more like the draftsman's work, not the architects.

The Mathematical Fraternity

The mathematical fraternity is a little like a self-perpetuating priesthood. The mathematicians of today train the mathematicians of tomorrow and, in effect, decide whom to admit to the priesthood. Most people do not find it easy to join--mathematical talent and genius are apparently exactly as rare as talent and genius in painting and music--but anyone can join, everyone is welcome. The rules are nowhere explicitly formulated, but they are intuitively felt by everyone in the profession. Mistakes are forgiven and so is obscure exposition--the indispensable requisite is mathematical insight. Sloppy thinking, verbosity without content, and polemic have no role, and--this to me is one of the most wonderful aspects of mathematics--they are much easier to spot than in the non-mathematical fields of human endeavor (much easier than, for instance, in literature among the arts, in art criticism among the humanities, and in your favorite abomination among the social sciences).

Although most of mathematical creation is done by one man at a desk, at a blackboard, or taking a walk, or, sometimes, by two men in conversation, mathematics is nevertheless a sociable science. The creator needs stimulation while he is creating and he needs an audience after he has created. Mathematics is a sociable science in the sense that I don't think it can be done by one man on a desert island (except for a very short time), but it is not a mob science, it is not a team science. A theorem is not a pyramid; inspiration has never been known to descend on a committee. A great theorem can no more be obtained by a "project" approach than a great painting; I don't think a team of little Gausses could have obtained the theorem about regular polygons under the leadership of a rear admiral anymore than a team of little Shakespeares could have written *Hamlet* under such conditions.

A Tiny and Trivial Mathematical Problem

I have been trying to give you a description of what mathematics is and how mathematicians do it, in broad general terms, and I wouldn't blame you if you had been finding it thoroughly unsatisfactory. I feel a little as if I had been describing snow to a Fiji Islander. If I told him snow was white like an egg, wet like mud, and cold like a mountain waterfall, would he then understand what it's like to ski in the Alps? To show, him a spoonful of scrapings from the just defrosted refrigerator of His Excellency the Governor is not much more satisfactory--but it is a little. Let me, therefore, conclude this particular tack by mentioning a tiny and trivial mathematical problem and describing its solution-possibly you'll then get (if you don't already have) a little feeling for what attracts and amuses mathematicians and what is the nature of the inspiration I have been talking about.

Imagine a society of 1025 tennis players. The mathematically minded ones among you, if you haven't already heard about this famous problem, have immediately been alerted by the number. It is known to anyone who ever kept on doubling something, anything, that 1024 is $2^{10.}$ All cognoscenti know, therefore, that the presence in the statement of a problem of a number like $1 + 2^{10}$ is bound to be a strong hint to its solution; the chances are, and this can be guessed even before the statement of the problem is complete, that the solution will depend on doubling--or halving--something ten times. The more knowledgeable cognoscenti will also admit the possibility that the number is not a hint but a trap. Imagine then that the tennis players are about to conduct a gigantic tournament, in the following manner. They draw lots to pair off as far as they can, the odd man sits out the first round, and the paired players play their matches. In the second round only the winners of the first round participate, and the whilom odd man. The procedure is the same for the second round as for the first--pair off and play at random, with the new odd man (if any) waiting it out. The rules demand that this procedure be continued, over and over again, until the champion of the society is selected. The champion, in this sense, didn't exactly beat everyone else, but he can say, of each of his fellow players, that he bet some one, who beat some one, who beat some one, who beat that player. The question is: how many matches were played altogether, in all the rounds of the whole tournament?

There are several ways of attacking the problem, and even the most naive one works. According to it, the first round has 512 matches (since 1025 is odd and 512 is a half of 1024), the second round has 256 (since the 512 winners in the first round, together with the odd man of that round, make 513; which is odd again, and 256 is a half of 512), etc. The "etcetera" yields, after 512 and 256, the numbers 128, 64, 32, 16, 8, 4, 2, 1, and 1 (the very last round, consisting of only one match, is the only one where there is no odd man), and all that is necessary is to add them up. That's a simple job that pencil and paper can accomplish in a few seconds; the answer (and hence the solution of the problem) is 1024.

The mathematical wiseacre would proceed a little differently. He would quickly recognize, as advertised, that the problem has to do with repeated halvings, so that the numbers to be added up are the successive powers of 2, from the ninth down to the first,--no, from ninth down to the zeroth!--together with the last 1 caused by the obviously malicious attempt of the problem-setter to confuse the problem-solver by using 1025 instead of 1024. The wiseacre would then proudly exhibit his knowledge of the formula for the sum of a geometric progression, he would therefore know (without addition) that the sum of 512, 256, ... , 8, 4, 2, and 1 is 1023, and he would then add the odd 1 to get the same total of 1024.

The trouble with the wiseacre's solution is that it's much too special. If the number of tennis players had been 1000 instead of 1025, the wise-acre would be no better off than the naive layman. The wiseacre's solution works, but it is as free of inspiration as the layman's. It is shorter but it is still, in the mathematician's contemptuous word, computational.

The problem has also an inspired solution, that requires no computation, no formulas, no numbers--just pure thought. Reason like this: each match has a winner and a loser. A loser cannot participate in any later rounds; every' one in the society, except only the champion, loses exactly one match. There are, therefore, exactly as many matches as there are losers, and, consequently, the number of matches is exactly one less than the membership of the society. If the number of members is 1025, the answer is 1024. If the number had been 1000, the answer would be *999,* and, obviously, the present pure thought method gives the answer, with no computation, for every possible number of players.

That's it: that's what I offer as a microcosmic example of a pretty piece of mathematics. The example is bad because, after all my warning that mathematicians are interested in other things than counting, it deals with counting; it's bad because it does not, cannot, exhibit any of the conceptual power and intellectual technique of non-trivial mathematics; and it's bad because it illustrates applied mathematics (that is, mathematics as applied to a "real life" problem) more than it illustrates pure mathematics (that is, the distilled form of a question about the logical interrelations of concepts--

concepts, not tennis players, and tournaments, and matches). For an example, for a parable, it does pretty well nevertheless; if your imagination is good enough mentally to reconstruct mathematics from the problem of the tennis players.

Mathology vs. Mathophysics

I've been describing mathematics, but, the truth to tell, I've had mathology (pure) in mind, more than mathophysics (applied). For some reason the practitioners of mathophysics tend to minimalize the differences between the two subjects and the others, the mathologists, tend to emphasize them. You've long ago found me out, I am sure. Every mathematician is in one camp or the other (well, almost every--a few are in both camps), and I am a mathologist by birth and training. But in a report such as this one, I must try not to exaggerate my prejudices, so I'll begin by saying that the similarities between mathology and mathophysics are great indeed. It is a historical fact that ultimately all mathematics comes to us, is suggested to us, by the physical universe: in that sense all mathematics is applied. It is, I believe, a psychological fact that even the purest of the pure among us is just a wee bit thrilled when his thoughts make a new and unexpected contact with the non-mathematical universe. The kind of talent required to be good in mathology is intimately related to the kind that mathophysics demands. The articles that mathophysicists write are frequently indistinguishable from those of the mathological colleagues.

As I see it, **the main difference between mathophysics and mathology is the *purpose* of the intellectual curiosity that motivated the work--or, perhaps, it would be more accurate to say that it is the *kind* of intellectual curiosity that is relevant**. Let me ask you a peculiar but definitely mathematical question. Can you load a pair of dice so that all possible rolls--better: all possible sums that can show on one roll, all the numbers between 2 and 12 inclusive--are equally likely? The question is a legitimate piece of mathematics; the answer to it is known, and it is not trivial. I mention it here so that you may perform a quick do-it-yourself psychoanalyst on yourself. When I asked the question, did you think of homogenous and non-homogeneous distributions of mass spread around in curious ways through two cubes, or did you think of sums of products of twelve numbers (the twice six possibilities associated with the twice six faces of the two dice)? If the former, you are a crypto-mathophysicist, if the latter you are a potential mathologist.

How do you choose your research problem, and what about it attracts you? Do you want to know about nature or about logic? Do you prefer concrete facts to abstract relations? If it's nature you want to study, if the concrete has the greater appeal, then you are a mathophysicist. In mathophysics the question always comes from outside, from the "real world," and the satisfaction the scientist gets from the solution comes, to a large extent, from the light it throws on *facts.*

Surely no one can object to mathophysics or think less of it for that; and yet many do. I did not mean to identify "concrete" with "practical" and thereby belittle it, and equally I did not mean to identify "abstract" with "useless." (That $2^{11213} - 1$ is a prime is a concrete fact, but surely a useless one; that $E = mc^2$ is an abstract relation but unfortunately a practical one.) Nevertheless, such identifications--applied-concrete-practical-crude and pure-abstract-pedantic-useless--are quite common in both camps. To the applied mathematician, the antonym of "applied" is "worthless," and to the pure mathematician the antonym of "pure" is "dirty."

History doesn't help the confusion. Historically, pure and applied mathematics (mathology and mathophysics have been much closer together than they are today. By now the very terminology (pure mathematics versus applied mathematics) makes for semantic confusion: it implies identity with small differences, instead of diversity with important connections.

From the difference in purposes follows a difference in tastes and hence of value judgments. The **mathophysicist wants to know the facts**, and he has, sometimes at any rate, no patience for the hair-splitting pedantry of the mathologist's rigor (which he derides as *rigor mortis).* The **mathologist wants to understand the idea**s, and he places **great value on the aesthetic aspects of the understanding** and the way that understanding is arrived at. He uses words such as "elegant" to describe a proof. In motivation, in purpose, frequently in method, and almost always in taste, the mathophysicist and the mathologist differ.

When I tell you that I am a mathologist, I am not trying to defend useless knowledge, or convert you to the view that it's the best kind. I would, however, be less than honest with you if I didn't tell you that I believe that. I like the idea of things being done for their own sake. I like it in music, I like it in the crafts, and I like it even in medicine. I never quite trust a doctor who says that he chose his profession out of a desire to benefit humanity; I am uncomfortable and skeptical when I hear such things. I much prefer the doctor to say that he became one because he liked the idea, because he thought he would be good at it, or even because he got good grades in high school zoology. I like the subject for its own sake, in medicine as much as in music; and I like it in mathematics.

Let me digress for a moment to a brief and perhaps apocryphal story about David Hilbert, probably the greatest mathematician of both the nineteenth and the twentieth centuries. When he was preparing a public address, Hilbert was asked to include a reference to the conflict (even then!) between pure and applied mathematics, with the hope that if anyone could take a step toward resolving it, he could. Obediently, he is said to have begun his address by saying "I was asked to speak about the conflict between pure and applied mathematics. I am glad to do so, because it is, indeed, a lot of nonsense--there should be no conflict, there can be no conflict--there is no conflict--in fact the two have nothing whatsoever to do with one another!"

It is, I think, undeniable that a great part of mathematics was born, and lives in respect and admiration, for no other reason than that it is interesting--it is interesting in itself. The angle trisection of the Greeks, the celebrated four-color map problem, and Godel's spectacular contribution to mathematical logic are good because they are beautiful, because they are surprising, because we want to know. ·Don't all of us feel the irresistible pull of the puzzle? Is there really something wrong with saying that mathematics is a glorious creation of the human spirit and deserves to live even in the absence of any practical application?

Mathematics is a Language

Why does mathematics occupy such an isolated position in the intellectual firmament? Why is it good form, for intellectuals, to shudder and announce that they can't bear it, or, at the very least, to giggle and announce that they never could understand it? One reason, perhaps, is that mathematics is a language. Mathematics is a precise and subtle language designed to express certain kinds of ideas more briefly, more accurately, and more usefully than ordinary language. I do not mean here that mathematicians, like members of all other professional cliques, use jargon. They do, at times, and they don't most often, but that's a personal phenomenon, not the professional one I am describing. What I do mean by saying that mathematics is a language is sketchily and inadequately illustrated by the difference between the following two sentences: 1) If each of two numbers is multiplied by itself, the difference of the two results is the same as the product of the sum of the two given numbers by their difference. 2) $x' - y' = (x + y) (x - y)$. (Note: the longer formulation is not only awkward, it is also incomplete.

One thing that sometimes upsets and repels the layman is the terminology that mathematicians employ. Mathematical words are intended merely as labels, sometimes suggestive, possibly facetious, but always precisely defined; their everyday connotations must be steadfastly ignored. Just as nobody nowadays infers from the name Fitzgerald that its bearer is the illegitimate son of Gerald, a number that is called irrational must not be thought unreasonable; just as a dramatic poem called *The Divine Comedy* is not necessarily funny, a number called imaginary has the same kind of mathematical existence as any other. (Rational, for numbers, refers not to the Latin *ratio,* in the sense of reason, but to the English "ratio," in the sense of quotient.)

Mathematics is a language. None of us feels insulted when a sinologist uses Chinese phrases, and we are resigned to living without Chinese, or else spend, years learning it. Our attitude to mathematics should be the same. **It's a language, and it takes years to learn to speak it well**. We all speak it a little, just because some of it is in the air all the time, but we speak it with an accent and frequently inaccurately; most of us speak it, say, about as well as one who can only say "*Oui, monsieur*" and "*S'il vous plaît*" speaks French. The mathematician sees nothing wrong with this as long as he's not upbraided by the rest of the intellectual community for keeping secrets. It took him a long time to learn his language, and he doesn't look down on the friend who, never having studied it, doesn't speak it. It is, however, sometimes difficult to keep one's temper with the cocktail party acquaintance who demands that he be taught the language between drinks and who regards failure or refusal to do so as sure signs of stupidity or snobbishness.

Some Analogies

A little feeling for the nature of mathematics and mathematical thinking can be got by the comparison with chess. The analogy, like all analogies, is imperfect, but it is illuminating just the same. The rules for chess are as arbitrary as the axioms of mathematics sometimes seem to be, The game of chess is as abstract as mathematics, (That chess is played with solid pieces, made of wood, or plastic, or glass, is not an intrinsic feature of the game. It can just as well be played with pencil and paper, as mathematics is, or blindfold, as mathematics can.) Chess also has its elaborate technical language, and chess is completely deterministic. There is also some analogy between mathematics and music. The mathologist feels the need to justify pure mathematics exactly as little as the musician feels the need to justify music. Do practical men, the men who meet payrolls, demand only practical music--soothing jazz to make an assembly line worker turn nuts quicker, or stirring marches to make a soldier kill with more enthusiasm? No, surely none of us believes in that kind of justification; music, and mathematics, are of human value because human beings feel they are.

The analogy with music can be stretched a little further. Before a performer's artistic contribution is judged, it is taken for granted that he hits the right notes, but merely hitting the right notes doesn't make him a musician. We don't get the point of painting if we compliment the nude Maya on being a good likeness, and we don't get the point of a historian's work if all we can say is that he didn't tell lies. Mere accuracy in performance, resemblance in appearance, and truth in storytelling doesn't make good music, painting, history: in the same way, mere logical correctness doesn't make good mathematics.

Goodness, high quality, are judged on grounds more important than validity, but less describable. A good piece of mathematics is connected with much other mathematics; it is new without being silly (think of a "new" western movie in which the names and the costumes are changed, but the plot isn't), alnd it is deep in an ineffable but inescapable sense--the sense in which Johann Sebastian is deep and Carl Philip Emmanuel is not. The criterion for quality is beauty, intricacy, neatness, elegance, satisfaction, appropriateness--all subjective, but all somehow mysteriously shared by all.

Mathematics resembles literature also, differently from the way it resembles music. The writing and reading of literature are related to the writing and reading of newspapers, advertisements, and road signs the way mathematics its is related to practical arithmetic. We all need to read and write and figure for daily lift,: but literature is more than reading and writing, and math is more than figuring. The literature analogy can be used to help understand the role of teachers and the role of the pure-applied dualism.

Many whose interests are in language, in the structure, in the history, and in the aesthetics of it, earn their bread and butter by teaching the rudiments of language to its future practical users. Similarly many, perhaps most, whose interests are in the mathematics of today, earn their bread and butter by teaching arithmetic, trigonometry, or calculus. This is sound economics: society abstractly and impersonally is willing to subsidize pure language and pure mathematics, but not very far. Let the would-be purist pull his weight by teaching the next generation the applied aspects of his craft; then he is permittted to spend a fraction of his time doing what he prefers. From the point of view of what a good teacher must be, this is good. A teacher must know more than the bare minimum he must teach; he must know more in order to avoid more and more mistakes, to avoid the perpetuation of misunderstanding, to avoid catastrophic educational inefficiency. To keep him alive, to keep him from drying up, his interest in syntax, his burrowing in etymology, or his dabbling in poetry playa necessary role.

The pure-applied dualism exists ill literature too. The source of literature is human life, but literature is not the life it comes from, and writing with a grim purpose is not literature. Sure there are borderline cases: is Upton Sinclair's *Jungle* literature or propaganda? (For that matter, is Chiquita Banana an advertising jingle or charming light opera?) But the fuzzy boundary doesn't alter the fact that in literature (as in mathematics) the pure and the applied are different in intent, in method, and in criterion of success.

Perhaps the closest analogy is between mathematics and painting. The origin of painting is physical reality, and so is the origin of mathematics--but the painter is not a camera, and the mathematician is not an engineer. The painter of "Uncle Sam Wants You" got his reward from patriotism, from increased enlistments, from winning the war--which is probably different from the reward Rembrandt got from a finished work. How close to reality painting (and mathematics) should be is a delicate matter of judgment. Asking a painter to "tell a concrete story" is like asking a mathematician to "solve a real problem." Modern painting and modern mathematics are far out--too far in the judgment of some. Perhaps the ideal is to have a spice of reality always present, but not to crowd it the way descriptive geometry, say, does in mathematics, and medical illustration, say, does in painting.

Talk to a painter (I did) and talk to a mathematician, and you'll be amazed at how similarly they react. Almost every aspect of the life and of the art of a mathematician has its counterpart in painting, and vice versa. Every time a mathematician hears "I could never make my checkbook balance" a painter hears "I could never draw a straight line"--and the comments are equally relevant and equally interesting. The invention of perspective gave the painter a useful technique, as did the invention 0 to the mathematician. Old art is as good as new; old mathematics is as good as new. Tastes change, to be sure, in both subjects, but a twentieth century painter has sympathy for cave paintings and a twentieth century mathematician for the fraction juggling of the Babylonians. A painting must be painted and then looked at; a theorem must be printed and then read. The painter who thinks good pictures, and the mathematician who dreams beautiful theorems are dilettantes; an unseen work of art is incomplete. In painting and in mathematics there are some objective standards of good--the painter speaks of structure, line, shape, and texture, where the mathematician speaks of truth, validity, novelty, generality--but they are relatively the easiest to satisfy. Both painters and mathematicians debate among themselves whether these objective standards should even be told to the young--the beginner may misunderstand and overemphasize them and at the same time lose sight of the more important subjective standards of goodness. Painting and mathematics have a history, a tradition, a growth. Students, in both subjects, tend to flock to the newest but, except the very best, miss the point; they lack the vitality of what they imitate, because, among other reasons, they lack the experience based on the traditions of the subject.

I've been talking *about* mathematics, but not *in* it, and, consequently, what I've been saying is not capable of proof in the mathematical sense of the word. I hope just the same, that I've shown you that there is a subject called mathematics (mathology?), and that that subject is a creative art. It is a creative art because mathematicians create beautiful new concepts; it is a creative art because mathematicians live, act, and think like artists; and it is a creative art because mathematicians regard it so. I feel strongly about that, and I am grateful for this opportunity to tell you about it. Thank you for listening.